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Simple analytical approximations to the integrals of the Bessel functions J_ν: application to the transmittance of a circular aperture

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Abstract

Two accurate, yet simple, analytic approximations to the integral of the Bessel function J_0 are presented. These first and second-order approximations are obtained by improving on the recently developed method known as *two-point quasi-rational approximations*. The accuracy of the first-order approximant is better than 0.05. The second-order approximant is practically indistinguishable from the true integral, even for very large values of the argument (overall accuracy is better than 0.002 05). Our approximants are, in addition, *analytic* and therefore replace with significant advantages both the well known power series and the asymptotic formulae of the integral. Approximants to the transmittance function of a plane wave through a circular aperture are derived, a problem which arises in diffraction theory and particle scattering. The second-order approximant to the transmittance is *analytic* too, and can be evaluated for small and large values of the argument, just with a hand-calculator. Its accuracy is better than 0.0011. As an extension, two first-order approximations to the integrals of the Bessel functions J_ν, of fractional order ν, are derived.

Mathematics Subject Classification: 65D20, 78A45

1. Introduction and summary

Bessel functions and integrals of Bessel functions often appear in several areas of physics and technology. A particular case, in diffraction theory, is the evaluation of the transmittance by a circular aperture which is given by the integral of the Bessel function $J_0(x)$. The numerical computation of this integral can certainly be done by summing its power series. However, the convergence of the series is extremely slow for very large values of the argument x. Even for medium values of the argument, $x = 5$ say, a large number of series terms are required to achieve good accuracy. The power series indeed become useless for not so large values of x. There occurs the important case of the region of intermediate values of the independent variable, where no simple numerical procedure exists for the computation of the integral. To solve this problem, approximations to the integral of J_0, which could be reliably used *for any* x, are derived here. The additional condition of *analyticity* is imposed on the new approximations. Note that rational approximations and *Padé approximants* [1–3] have been widely used for the evaluation of special functions in many cases of practical interest in physics and engineering. However, in the case of the integral of J_0 they do not provide a good approximant, one valid for any value of x, even if high-degree polynomials are used. On the other hand, a new so-called *quasi-rational approximations* method has been developed [4–6] which simultaneously exploits both the power and the asymptotic expansions. Such a method blends rational functions with other non-rational functions such as trigonometric and power functions, i.e. a true quasi-rational approximation method. It does have appreciable advantages, among them being acceptable accuracy and quick numerical evaluation. In a recent work [4] the method was used to generate a quasi-rational approximant for the Bessel function J_0. Although this last approximant is integrable in closed form, the calculation is still laborious. It became clear that a *direct* approximation to the integral of J_0 itself would be of greater interest, and clearly more useful. The advantages of such a type of approximant would be appreciated in applied optics, fibre optics and particle scattering theory. For instance, the problem of wave diffraction through a circular aperture leads to the integral of the Bessel function J_0; and the same happens when the transmittance of a circular aperture is evaluated in the usual Kirchhoff vectorial theory of diffraction [7–9].

In this paper we develop *direct quasi-rational* approximations, of first and second degree, for the evaluation of the transmittance through a circular aperture of a normally incident plane wave; the second-degree approximant being the more accurate. The term 'direct' means that the transmittance is to be obtained directly by approximating the integral of the Bessel function J_0 function itself, instead of first obtaining the approximation to J_0, and then proceeding to integrate it in a second step. It will be seen that our approximants are rather simple, *fast*, short and easy to evaluate, even with a hand-calculator. Furthermore, their accuracy is very good for any application and, in addition, valid along the *whole* positive real axis, i.e. no need to separate into two (finite and asymptotic cases) recurrent computer algorithms. It is known that the integral of J_0 can be evaluated with high accuracy using available computational techniques (for instance *Chebyshev coefficients* [10] have been determined for the integrals of J_0 in the separate intervals $(0, x_0)$ and (x_1, ∞)). However, the need for simple, short approximants which can be quickly dealt with using just a hand-calculator, *for any* x, has not been simply satisfied. Besides, the approximants here derived are *analytic* and may therefore replace the *exact* integral in algebraic expressions, granting analytical differentiation and integration. This is an additional and truly important asset of such approximants.

The method to find approximations exploited here is by no means exclusive to the Bessel function J_0. As an extension, a couple of first-order approximations to the fractional order integrals of the first-kind Bessel function, namely, $\int_0^x J_\nu(t)\,\mathrm{d}t$, $\nu > -1$, are also derived. In order to obtain good accuracy the form of the approximation for positive fractional order has to be different from that for negative order. In both cases the coefficients of the approximants are functions of the order ν. For each approximant the first terms of the power series, as well as those of the asymptotic expansion, of the integral are equated with corresponding terms of the approximant. The equations thus obtained lead to the approximant coefficients. The procedure is described below in detail (sections 2 and 5).

2. Approximation procedure

Consider the problem of evaluating the integral

$$I(x) = \int_0^x J_0(t)\,\mathrm{d}t \tag{1}$$

with the known boundary condition $I(\infty) \to 1$. Firstly, note that in order to approximate the function I with the quasi-rational method, its expansions around two points (at the origin and at infinity) must be known. Both the power series and the asymptotic expansion for $I(x)$ are therefore required. The power series for $I(x)$ is easily obtained by straightforward integration of the power series [11] for J_0,

$$I(x) = x - \frac{x^3}{12} + \frac{x^5}{320} - \frac{x^7}{16\,128} \cdots \tag{2}$$

while the asymptotic expansion [11] is given by

$$I(x) \approx 1 - \frac{1}{\sqrt{\pi x}} \left\{ \left[1 + \frac{5}{8x} + \mathrm{O}(1/x^2) \right] \cos x + \left[-1 + \frac{5}{8x} + \mathrm{O}(1/x^2) \right] \sin x \right\}. \tag{3}$$

A quasi-rational approximation to the function I given in equation (1) is sought taking into account both expansions (2) and (3). The main criterion is that the approximant, here denoted by A_n, should have the same singularities as the integral I, in the region of interest. Thus the form of the approximants A_n must be written as

$$A_n(x) = 1 + \frac{1}{\sqrt{1 + \mu^2 x}} \frac{\left[\left(\sum_{i=0}^n P_i x^i \right) \cos x + \left(\sum_{i=0}^n p_i x^i \right) \sin x \right]}{\sum_{i=0}^n q_i x^i} \tag{4}$$

where n is the order of the approximation, and μ is a *free parameter* whose important role, and the way to prescribe it, will be discussed presently.

Note that in equation (4) the exponent of the factor $1/(1 + \mu^2 x)$ has to be $\frac{1}{2}$. It is written in this way because for *large values of x* the approximation should have the same behaviour as $(\sin x)/\sqrt{x}$ as well as that of $(\cos x)/\sqrt{x}$, a behaviour characteristic of the asymptotic expansion of the integral $I(x)$.

The q, p and P coefficients in equation (4) are determined by choosing a suitable number of terms from the power and asymptotic expansions. Some care, however, is required in choosing them. Acceptable accuracy will be achieved only if a procedure to avoid the so-called *approximation defects* [1, p 53] is available and applied. Fortunately, a good way to overcome such potential difficulties is to exploit the presence of the additional *free parameter* μ in equation (4). The coefficients of the approximant can then be determined as functions of μ. In this way, a real value for μ can be simply chosen, such that the *defects* are avoided. Clearly, a better way to enjoin the value of μ could be, for instance, to look for the least maximum absolute error of the approximant. This is precisely the procedure followed below. Other optimization criteria may be used, e.g. the least quadratic distance, or the least maximum relative error. Note that these ways of prescribing the value of this free parameter μ rely on former knowledge of the function being approximated. This, of course, is not always the case, and in section 5 we describe an analytic method to find the best μ.

Given the simple form of the first-order quasi-rational approximation, its derivation begins by defining the form of the denominator, and the parameter μ is chosen to be just equal to one. On the other hand, in the case of the second-order approximant we determine all the coefficients, P, p and q, as functions of the free parameter μ, and then the latter is later found by optimization as already mentioned above.

3. Numerical results

Consider now the first-order approximant A_1 to the integral of J_0. Set $n = 1$ in equation (4). The two coefficients, P_1 and p_1, are determined by the leading term of the asymptotic expansion for I (equation (3)). The results are as follows:

$$P_1 = -\frac{1}{\sqrt{\pi}} \qquad p_1 = \frac{1}{\sqrt{\pi}}. \tag{5}$$

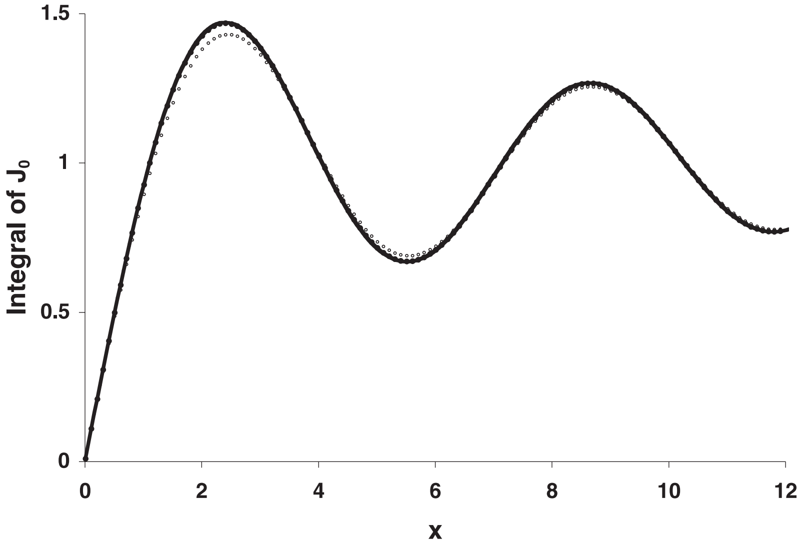

Figure 1. Graph of the integral of the Bessel function J_0 (full curve) in the interval (0, 12). The first-order direct two-point approximant (open circles) and the second-order approximant (dotted curve) to the integral have been also plotted for comparison. The graph of the second-order approximant is indistinguishable because of its very good accuracy (even for very large values of the arguments).

Using these values, the first-order approximation can be readily written as

$$A_1(x) = 1 + \frac{-\left(1 + x/\sqrt{\pi}\right)\cos x + \left(p_0 + x/\sqrt{\pi}\right)\sin x}{(1+x)^{3/2}} \approx \int_0^x J_0(t)\,\mathrm{d}t. \tag{6}$$

Here $P_0 = -1$ in order to satisfy the well known boundary condition of the integral of the Bessel function J_0: $I_1(x) = 0$ at $x = 0$. To determine the remaining unknown coefficient p_0 the functions $\cos x$, $\sin x$ and $(1+x)^{3/2}$ are replaced in equation (6), by the corresponding power series of the functions, and the result is then compared with the power series given in equation (2). Equating the coefficients of first-degree terms in x, one immediately obtains the required value of p_0,

$$p_0 = \frac{\left(2 - \sqrt{\pi}\right)}{2\sqrt{\pi}}. \tag{7}$$

The first-order approximant A_1 to the integral of the Bessel function J_0 is therefore the rather simple relation

$$A_1(x) = 1 + \frac{\left(1 - \sqrt{\pi}/2 + x\right)\sin x - \left(\sqrt{\pi} + x\right)\cos x}{\sqrt{\pi}(1+x)^{3/2}} \tag{8}$$

easily calculable with just a hand-calculator for any $x \in R$. This expression gives the values of the integral in equation (1) with a maximum error $\varepsilon = 0.047$, which occurs at about $x = 1.8$. It may, therefore, be used to obtain results quickly, when one does not require very high accuracy. The first-order approximant is plotted in figure 1, where for the sake of comparison the exact integral of the Bessel function J_0 has been plotted too. Note the high accuracy of this simple approximant for $x > 3$.

Here the error ε has been found by a simple comparison of the approximant values with those given by the exact function. This is, in fact, the best way to find the error. However, in the case where the exact function values cannot be calculated, a theoretical bound for the error can be found by an analytical procedure described in [5]. Following this procedure the next terms of both the power series and the asymptotic expansion of the difference $A_1(x)$–$I(x)$ have to be found, which in the present case leads to

$$A_1(x) - I(x) = \left(\frac{1}{\sqrt{\pi}} - \frac{5}{8}\right)x^2 + \mathrm{O}(x^3) = -0.051x^2 + \mathrm{O}(x^3)$$

for the power series, and to

$$A_1(x) - I(x) \approx \frac{1}{\sqrt{\pi x}}\left\{\left[\left(\frac{1}{\sqrt{\pi}} - \frac{5}{8}\right)\frac{1}{x} + \mathrm{O}(x^3)\right]\sin x + \left[\left(\tfrac{17}{8} - \sqrt{\pi}\right)\frac{1}{x} + \mathrm{O}(x^3)\right]\cos x\right\}$$

for the asymptotic expansion. Looking for the upper bound for the error, the abscissa x_0 at which the maximum theoretical error ε_1 occurs has to be determined [5]. From the two previous equations one obtains

$$0.051x_0^2 = (0.430 + 0.200)\frac{1}{x_0^{3/2}} \quad \Leftrightarrow \quad x_0 = 1.840.$$

The theoretical bound for the error [5] is then $\varepsilon_1 = 0.051(1.840)^2 = 0.173$. This is about three times as large as the error found above ($\varepsilon = 0.047$) by direct comparison of the approximant with the exact function. As discussed in [5] the theoretical bound for the error is usually larger that the actual error. Therefore, *when possible* it is better to find the errors using the numerically calculated values of the exact function.

Consider now the derivation of a second-degree direct approximation to the integral of the Bessel function J_0. Using a procedure analogous to the one described above for the first-degree approximant, one begins by writing A_2,

$$A_2(x) = 1 + \frac{(P_0 + P_1x + P_2x^2)\cos x + (p_0 + p_1x + p_2x^2)\sin x}{\sqrt{\pi}(1 + \mu^2x)^{1/2}(q_0 + q_1x + q_2x^2)}. \tag{9}$$

For the sake of simplicity one begins by setting $q_0 = 1$ in equation (9). Also, to recover the boundary zero value, of the integral at $x = 0$, one must set $P_0 = -\sqrt{\pi}$ in equation (9). Note that one can also prescribe the coefficients p_2, P_2 of the second-order terms in the numerator of the approximant. In effect, for large values of the argument x, and from a straightforward comparison of the leading terms of the coefficients of $\cos x$ and $\sin x$, which appear in the asymptotic formula (3), with the corresponding terms of our approximant (9), one obtains

$$\frac{1}{\sqrt{\pi x}}\sin x \cong \frac{p_2x^2}{\sqrt{\pi}\mu\sqrt{x}q_2x^2}\sin x \quad \Rightarrow \quad p_2 = \mu q_2$$

and

$$-\frac{1}{\sqrt{\pi x}}\cos x \cong \frac{P_2x^2}{\sqrt{\pi}\mu\sqrt{x}q_2x^2}\cos x \quad \Rightarrow \quad P_2 = -\mu q_2.$$

To determine the remaining coefficients in equation (9), one has to use the two leading terms of the asymptotic expansion (3), and one term each from the sine function and the cosine function. Then equate the coefficients of the terms of the expression obtained (up to third order x^3) with the corresponding terms of the power series (2). After some lengthy, but, in fact, easy, algebraic work, one obtains expressions for the whole set of coefficients of the sought approximant, namely,

$$P_0 = -\sqrt{\pi} \qquad P_1 = -\mu\left[q_1 + q_2\left(\frac{1}{2\mu^2} + \frac{5}{8}\right)\right] \qquad P_2 = -\mu q_2 \tag{10a}$$

$$\begin{aligned} p_0 &= q_1\left(-\sqrt{\pi} + \mu\right) + q_2\left(\frac{1}{2\mu} + \frac{5}{8}\mu\right) - \sqrt{\pi}\left(\frac{\mu^2}{2} - 1\right) \\ p_1 &= \mu\left[q_2\left(-\frac{5}{8} + \frac{1}{2\mu^2}\right) + q_1\right] \end{aligned} \tag{10b}$$

$$p_2 = \mu q_2 \qquad q_0 = 1 \qquad q_1 = \frac{\beta}{\alpha} q_2 + \frac{\gamma}{\alpha} \qquad q_2 = \left(\frac{b\gamma}{d\alpha} + \frac{c}{d}\right) \Big/ \left(1 - \frac{b\beta}{d\alpha}\right) \tag{10c}$$

where the real quantities d, b, c, α, β and γ are given by the following functions of the free parameter μ:

$$\begin{aligned} d &= \frac{\sqrt{\pi}}{2}\left(2 - \mu^2\right) - \frac{29}{24}\mu - \frac{1}{6\mu} & b &= \frac{\mu}{3} + \frac{\sqrt{\pi}}{2}\left(\frac{4 - 12\mu^2 - 3\mu^4}{12}\right) \\ c &= \frac{\sqrt{\pi}}{2}\left(\frac{-2 + 2\mu^2 + 3\mu^4 + \mu^6}{12}\right) & \alpha &= -\mu - \frac{\sqrt{\pi}}{2}\left(-2 + \mu^2\right) \\ \beta &= -\frac{13}{8}\mu + \frac{1}{2\mu^2} + \sqrt{\pi} & \gamma &= \frac{\sqrt{\pi}}{2}\left(\frac{4 - 4\mu^2 - \mu^4}{4}\right). \end{aligned} \tag{11}$$

Apart from q_0 and P_0, the remaining coefficients of the approximant are defined as functions of the free parameter μ. As already stated above, for some values of μ defects may arise, and this parameter is to be chosen with care. Yet there is ample freedom in the choice of μ, an advantage exploited here to improve the accuracy of the approximant.

Since, as mentioned before, there are several alternative ways of determining the best value for the parameter μ, it is useful to discuss the matter in some detail. The procedure described below is general and can be applied to any special function, therefore it is better to discuss the point within a general framework. Should the special function be known *via* its power series or by its asymptotic expansion, then the best value of μ can be analytically determined using an additional term of the power series, or one more term from the asymptotic expansion. This will lead to an additional algebraic equation in the parameter μ and to the set of equations (10). In the present case it happened that the derivation of such equations was far easier when the additional term was taken from the asymptotic expansion, rather than from the power series. Taking then one term more of the asymptotic expansion of the integral of J_0 gives

$$\begin{aligned} I(x) \approx 1 - \frac{1}{\sqrt{\pi x}} \Bigg\{ &\left[1 + \frac{5}{8x} - \frac{129}{128x^2} + \mathrm{O}(1/x^3)\right] \cos x \\ &+ \left[-1 + \frac{5}{8x} + \frac{129}{128x^2} + \mathrm{O}(1/x^3)\right] \sin x \Bigg\}. \end{aligned} \tag{12}$$

The two second-order terms just added are readily obtained using the *coefficients recurrence* formula [11, equation (11.1.2)] for the asymptotic expansion of the integral. Since only one equation is needed to find μ the additional term to be considered could be either $-\frac{129}{128x^2}\sin x$, or $-\frac{129}{128x^2}\cos x$. Selecting here the first and equating with the expansions for the coefficients of $\sin x$ in the second-order approximant equation (9) one obtains

$$\begin{aligned} \mu q_2 &\left(1 - \frac{5}{8x} - \frac{129}{128x^2} + \cdots\right)\left(1 + \frac{q_1}{q_2 x} + \frac{1}{q_2 x^2}\right) \\ &= p_2\left(1 + \frac{p_1}{p_2 x} + \frac{p_0}{p_2 x^2}\right)\left(1 - \frac{1}{2\mu^2 x} + \frac{3}{8\mu^4 x^2} + \cdots\right). \end{aligned} \tag{13}$$

After a few additional algebraic steps the following equation ensues:

$$\frac{\sqrt{\pi}}{2}\mu^5 + \left(1 - \frac{13q_1}{8} - \frac{169}{128}\right)\mu^4 + \sqrt{\pi}(q_1 - 1)\mu^3 + \left(\frac{q_1}{2} - \frac{13}{16}\right)\mu^2 - \frac{1}{8} = 0 \tag{14}$$

with q_1 and q_2 given as in (10*c*) above. Solving equation (15) one obtains $\mu = 1.021\,471$, the sought value for the free parameter of the approximant.

Although the procedure just described should always give good results, it is clear the the best μ value is that one which gives the least minimum error. Therefore, should the values of the function being approximated be known for several points, μ could be obtained by minimizing the distance to those points. In some cases, for example, Bessel functions, we can use as many points as we like.

When we plot the maximum absolute error of our second-order approximant to the integral of the Bessel function J_0 *as a function of the parameter* μ, using equations (10) and (11), the optimum value $\mu = 1.02$ is found, which incidentally happens to correspond to a *cusp* in the corresponding function plot (not shown in this paper). The maximum absolute error of the second-order approximant is $\varepsilon = 0.002\,042$, for the optimum value $\mu = 1.02$. This value is very close indeed to the previous one $\mu = 1.021\,471$, found analytically with the procedure based on the addition of one more term to the asymptotic expansion described before.

After substituting the value $\mu = 1.02$ into equations (10) and (11), the P, p, q coefficients of approximant A_2 are found,

$$\begin{array}{lll} P_0 = -1.772\,4540 & P_1 = -1.160\,5640 & P_2 = -0.206\,6663 \\ p_0 = 0.391\,3175 & p_1 = 0.902\,2309 & p_2 = -P_2 \\ q_0 = 1 & q_1 = 0.913\,8008 & q_2 = 0.202\,6140. \end{array}$$

In figure 1, as well as plotting the first-order approximant, we have plotted our second-order approximant too. But now, because of the large improvement in accuracy obtained with our second approximation, the graph of the latter cannot easily be distinguished from the graph of the integral of the Bessel function J_0. Large scale-up factors have to be used to show the errors of the approximant. The two curves remain indistinguishable even for very large values of the argument. The maximum absolute error of the second-order direct approximation is found to be less than 0.002 05 (for the chosen $\mu = 1.02$). It occurs at the argument value $x \approx 2.01$. In comparison with the case of the first-order approximant, we note that the absolute error is reduced, actually divided by about 25. In figure 2 we plot the small errors (< 0.05) of the two approximants. Note that the error of the second approximant (the full curve) has to be plotted amplified by a factor of 10.

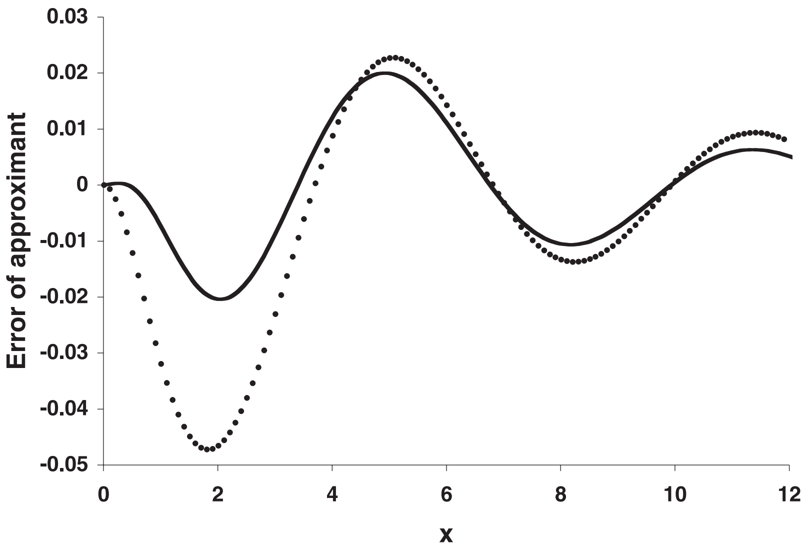

Figure 2. Small errors of the first- and second-order approximants to the integral of the Bessel function J_0. Note that for the second-order approximant (full curve) the error has to be amplified by 10. The dotted curve represents the error of the first-order approximant.

It is interesting to point out that when resorting to the inclusion of a free parameter in a given approximant, its accuracy could also be good even if μ is not exactly the optimum value, but relatively close to it. For instance, if instead of $\mu = 1.02$ the value $\mu = 1$ is substituted in the second-order approximant then the maximum error found is still only $\approx 5 \times 10^{-3}$, which is also a very good result. As far as μ is concerned the key point is always to choose a positive value in order to avoid the *defects* (an extraneous pole and a nearby zero) which are so common in the Padé approximants [1, pp 53–55], and that can also surface in our approximations.

4. An application

As an application the two approximants derived above are used to evaluate the transmittance T of a plane monochromatic wave by a circular aperture of known radius a, in an infinite plane conducting screen, under normal incidence. This is a standard problem in diffraction of electromagnetic waves and in particle scattering. The transmittance of such an aperture is given by

$$T(a) = 1 - \frac{I(2ka)}{2ka} \tag{15}$$

where k is the wavenumber of the incident wave. First- and second-order approximants to this transmittance are obtained when the two approximants A_1, A_2 are sequentially replaced in equation (15). The errors, grossly amplified, of the two resulting transmittance approximant functions appear plotted in figure 3 (the broken curve for the first-order approximant error is amplified by a factor of 1000; the full curve for the error of the second-order one after amplification by a factor of 10 000).

In a previous work [4] another approximant to the transmittance T was presented. It was obtained by integrating an approximant to the Bessel function J_0 itself, that is to say using an

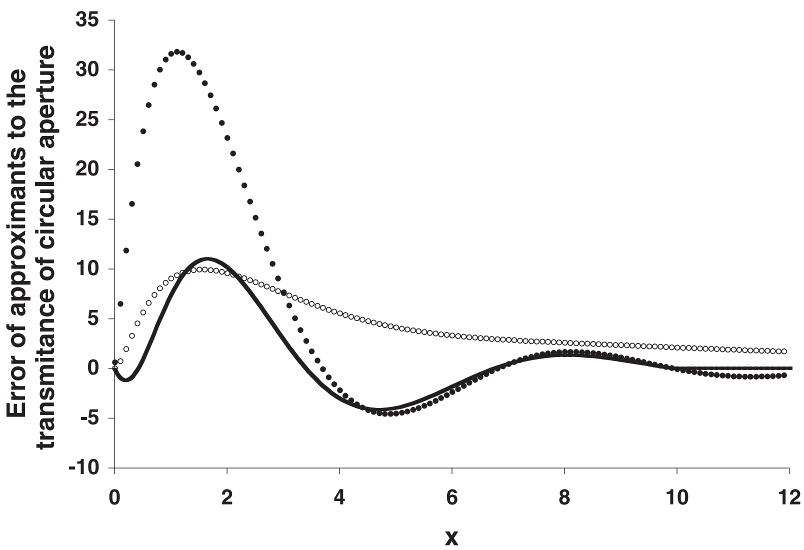

Figure 3. Amplified errors of the direct two-point quasi-rational approximants to the transmittance of a circular aperture. The full curve is the error, after amplification by a factor of 10 000, of our second-degree approximant. The dotted curve represents the error of the first-order approximant (present work) after amplification by 1000. For comparison, the error of an (indirect) approximation to the transmittance (from an earlier work) is also shown (open circles), amplified 1000 times.

indirect method (first an approximant to J_0 was obtained, then an approximant to the integral I, and finally the approximant to transmittance T). The error of this previous approximant is also plotted (broken curve) in figure 3 for the sake of comparison with the approximations of this paper.

5. Approximation to the integral of the Bessel function J_ν, $\nu > -1$

In this section the procedure presented in sections 2 and 3 is applied to obtain approximations to the function I_ν defined by

$$I_\nu(x) = \int_0^x J_\nu(t)\,\mathrm{d}t \qquad \nu > -1. \tag{16}$$

This integral tends to 1 in the limit $x \to \infty$ [11, equation (11.4.17)]. It is possible to obtain a power-series expansion for I_ν by integrating the known power series of J_ν [8, 11]. Moreover, the required asymptotic expansion of I_ν is obtained, assuming that it has a form similar to that of the asymptotic formula of J_ν, with as yet unknown coefficients. But those coefficients may be found by applying the well known *method of undetermined coefficients* to the derivative of I_ν and to J_ν itself. Therefore, one obtains the power series

$$I_\nu(x) = \frac{x^{\nu+1}}{2^\nu} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,(2j+\nu+1)\,\Gamma\,(j+\nu+1)} \left(\frac{x}{2}\right)^{2j} \qquad x \ll 1 \tag{17}$$

and the required asymptotic expansion

$$I_\nu(x) = 1 + \frac{\beta}{\sqrt{\pi x}} \left\{1 + \frac{4\nu^2-5}{8}\frac{\alpha}{\beta}\frac{1}{x} \pm \cdots\right\} \sin x - \frac{\alpha}{\sqrt{\pi x}} \left\{1 - \frac{4\nu^2-5}{8}\frac{\beta}{\alpha}\frac{1}{x} \pm \cdots\right\} \cos x \qquad x \gg 1 \tag{18}$$

where α and β are defined as

$$\alpha = \sqrt{2}\sin\left[\left(\frac{\nu}{2}+\frac{1}{4}\right)\pi\right] \qquad \beta = \sqrt{2}\cos\left[\left(\frac{\nu}{2}+\frac{1}{4}\right)\pi\right].$$

Two versions of the sought first-order approximant are examined here. These are denoted ${}^1I_\nu(x)$ and ${}^2I_\nu(x)$, respectively, and have the forms,

$${}^1I_\nu(x) = \frac{x^\nu}{(1+x^2)^{\nu/2}} \left[1 + \frac{(p_0+p_1x)\sin x + (P_0+P_1x)\cos x}{(1+x)^{3/2}}\right] \tag{19a}$$

$${}^2I_\nu(x) = \frac{x^{\nu+1}}{(1+x^4)^{\nu/4}} \left[1 + \frac{(p_0'+p_1'x)\sin x + (P_0'+P_1'x)\cos x}{(1+x)^{3/2}}\right]. \tag{19b}$$

The coefficients of the approximants are obtained by considering the $j = 0$ term in the power series and the two leading terms of the asymptotic expansions of $\sin x$ and $\cos x$. The condition that the integral of J_ν from zero to infinity is one is already ensured. Note that in the power-series expansion of the approximant denoted ${}^1I_\nu(x)$ there is a term in x^ν whose coefficient, in fact, must be zero, since such a term does not appear in the expression for the integral I_ν (equation (17)). Furthermore, note that the coefficient of the term in $x^{\nu+1}$ must be $2^{-\nu}(\Gamma(\nu+2))^{-1}$. Therefore, there are two equations from the power series and two more from the leading terms of the asymptotic expansion. In this way there are the same number of equations as unknowns and the four approximant parameters can be found. In the case of the other approximant ${}^2I_\nu(x)$ there is no power term in x^ν, then the coefficients of $x^{\nu+1}$ and $x^{\nu+2}$ must be used instead. Note that the coefficient of $x^{\nu+2}$ in the power series of I_ν is zero.

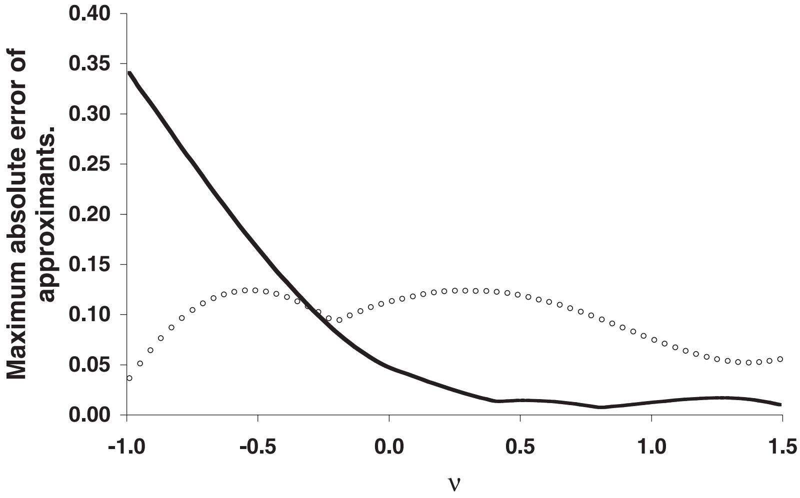

Figure 4. Absolute maximum errors of the two first-order approximants ${}^1I_\nu(x)$ and ${}^2I_\nu(x)$ to the integral of the Bessel function J_ν in the interval $x \in (0, 10)$ as a function of the fractional order $\nu \in (0, 10)$. The full curve represents the maximum absolute errors of the approximant ${}^1I_\nu(x)$. The broken curve corresponds to similar errors for ${}^2I_\nu(x)$.

After some steps all the approximant parameters (p_j and P_j) are found and their expressions ensue,

$$ {}^1I_\nu(x) = \frac{x^\nu}{(1+x^2)^{\nu/2}}\left\{1 + \left[\left(\alpha + \frac{\sqrt{\pi}}{2^\nu\Gamma(\nu+2)} - \frac{3\sqrt{\pi}}{2} + \beta x\right)\sin x - \left(\sqrt{\pi} + \alpha x\right)\cos x\right] \times\left[\sqrt{\pi}(1+x)^{3/2}\right]^{-1}\right\} \tag{20a} $$

$$ {}^2I_\nu(x) = \frac{x^{\nu+1}}{(1+x^2)^{(\nu+1)/4}}\left\{1 + \left[\left(\alpha + \frac{3\sqrt{\pi}}{2^{\nu+1}\Gamma(\nu+2)} - \frac{3\sqrt{\pi}}{2} + \beta x\right)\sin x + \left(\frac{\sqrt{\pi}}{2^{\nu+1}\Gamma(\nu+2)} - \sqrt{\pi} + \alpha x\right)\cos x\right]\left[\sqrt{\pi}(1+x)^{3/2}\right]^{-1}\right\}. \tag{20b} $$

In figure 4 we show the maximum absolute errors of the two approximants ${}^1I_\nu(x)$ and ${}^2I_\nu(x)$, as a function of fractional order $\nu \in (-1, 1.5)$. As expected the previous results are recovered, i.e. for $\nu = 0$ the new approximant ${}^1I_\nu(x)$ coincides with the approximant $A_1(x)$ given in equation (6). In figure 4 one sees that the approximant is very good for positive ν, while its maximum absolute error becomes large in the region where ν is negative (for $\nu < -0.5$). On the other hand, ${}^2I_\nu(x)$ is good for negative ν, particularly for $\nu \approx -1$, but not close to -0.5.

It is also relevant to examine the behaviour of the approximants for small, intermediate and large values of the argument x. In figure 5 the exact curves for I_ν and its approximant curves are shown for three values of the fractional order, $\nu = -\frac{1}{2}$, 0 and $\frac{1}{2}$. At the scale of the plot the approximant curves and the exact ones are almost coincident for $\nu = 0$ and $\frac{1}{2}$. As expected, errors are negligible for small and large values of the argument x. In the case of $\nu = -\frac{1}{2}$, the largest error occurs in the neighbourhood of $x = 1$. The truly important point to note is that the minima and maxima of the approximant occur at points very close to where the exact ones occur, which is the crucial fact for most applications.

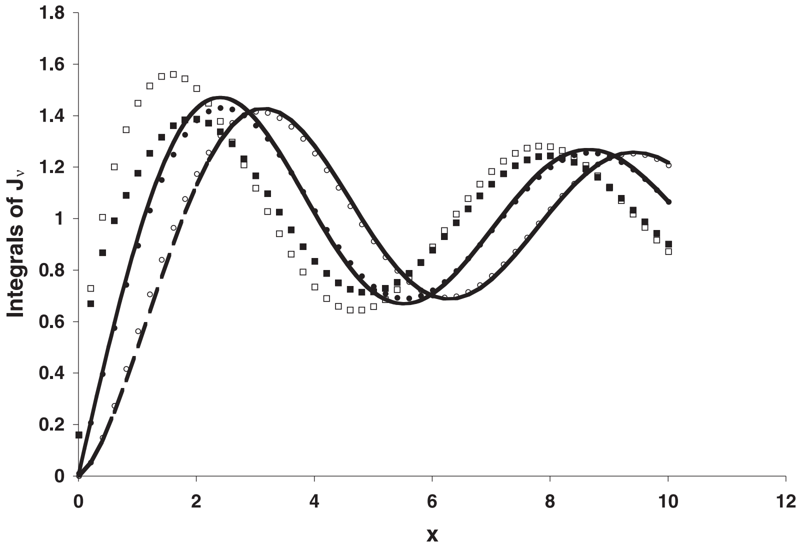

Figure 5. Exact function $I_\nu(x) = \int_0^x J_\nu(t)\,dt$ plotted versus the argument x, for three values of the fractional order $\nu = -\frac{1}{2}$, 0 and $\frac{1}{2}$ (full squares, full and broken curves, respectively). The curves for the three first-order approximants to the same integrals are also shown (open squares, dots and open circles, respectively) for the same values of ν.

6. Conclusions

In this work two new, simple and analytic, approximations to the integral of the Bessel function J_0 were derived. Apart from being easily calculable, the two approximants have good accuracy, particularly the second-order one. The maximum absolute departure of the latter from the true value of the integral is less than 0.002 05 and occurs at $x \approx 2$, just at the beginning of the region of intermediate values of the argument. The present approximants are very accurate for large values of the argument. The two approximants found have then been applied to obtain first- and second-order approximants to the coefficient of transmittance of a plane wave through a circular aperture of known radius. Maximum departures of these approximants from the exact transmittance values are less than 0.032 and 0.0012, respectively. The larger errors occur for intermediate values of the argument (between 1.5 and 5.5), and the errors are truly negligible for large values of the argument. As a comparison, the maximum error of the approximant in a previously published work [4] is about 0.01. The present second-order, direct quasi-rational, approximant is about 10 times better. In addition the new approximants have the advantages of being simpler, and a lot easier to evaluate. Moreover, being directly integrable and derivable, the new approximants have a larger potential for applications, for instance should further mathematical analysis be required in the applications.

Finally, note that with present (since 1994) commercially available computer software one can also evaluate the integral of the Bessel function J_0 using the power series and the asymptotic expansions. But again, two computer codes have to be written, and in addition their intervals of application have to be accurately estimated in advance for each particular application. As soon as the integral argument goes beyond medium values ($x = 6$, say), a very large number of terms of the power series are required to achieve acceptable accuracy. The second-order approximant presented in this work means: a single short code, manageable with even a hand-calculator, which is both derivable and integrable, and of very good accuracy from zero practically to infinity.

As an extension a couple of first-order approximants to the integral of the Bessel function of fractional order $\nu > -1$ have also been derived. The first of them (equation (20a)) has very good accuracy (maximum error of less than 0.05) for positive values of ν, and for $\nu = 0$. Its errors increase practically linearly (figure 4), and eventually become large for ν less than about -0.5. The second first-order approximant (equation (20b)) has acceptable accuracy for negative values of ν, except very close to $\nu = -\frac{1}{2}$. The simple first-order approximants to the general integral of J_ν do not have such excellent accuracy but they are still sufficient for a large number of applications.

It is also possible to derive higher-order approximations to the integral I_ν of higher accuracy, using the method described in this work. But already, for the lowest higher order, that is for the second order, the algebraic process to derive a general approximation, i.e. one valid for any ν, quickly becomes very difficult indeed. Second-order approximants to the integral I_ν have to be developed for each particular value of ν instead, as was done in this paper for the integral of the Bessel function J_0 (section 3).

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